Notes:

- Sections marked with **[Extra]** are optional for additional learning. These topics will not be included in your graded assessments.
- Sections marked with **[Self]** are required for independent study. You will be tested on these topics.

Chapter 1: Speaking Mathematically

1. **Sets**
    - The Set-Roster and Set-Builder Notations
    - Subsets:
    - Set Membership:
    - Examples of Sets
        -
    - Cartesian Products
    - **[Extra]** Strings

Relations and Functions

- Definition of a Relation
- Arrow Diagram of a Relation
- Examples of Relations
- Definition of Function
    - Domain
    - Codomain & Range
- Function Machines
- Equality of Functions
- Examples of Functions

Graphs

- Definition
- Undirected and Directed Graphs

- Representation
- Degree of a Vertex
- Examples of Graphs

Chapter 2: The Logic of Compound Statements

- **Logical Form and Logical Equivalence**:
    - Statements
    - Truth Values
    - Compound Statements
        - Negation (or   )
        - Conjunction (   )
        - Disjunction (   )
        - Exclusive-Or (   )
    - Convert English Sentences into Logical Form
    - Evaluating the Truth of General Compound Statements
        - Truth Tables
    - Logical Equivalence
        - Negation of Conjunction, Disjunction, and Negation
        - De Morgan's Laws
    - Tautologies and Contradictions
    - Summary of Logical Equivalences & Operations

- **Conjunctive Statements**
    - BUT as AND
- **Conditional Statements (Implications)**
    - Implication (→)
        - Hypothesis and Conclusion
    - Implication as Disjunction
    - The Negation of an Implication
    - Double Implication (   )
        - Logical Equivalency (   )
            - Not the same as Equality
    - The Contrapositive of a Conditional Statement (   )
    - The Converse of a Conditional Statement (   )

- The Inverse of a Conditional Statement (   )
- Conditional Statements Variations

    - *If P then Q* (   )

    - *P if Q* (   )

    - *P Only If Q* (   )

    - *P if and only if Q* (   )
- Necessary and Sufficient Conditions

Valid and Invalid Arguments

- Definition of Argument

    - Premises and Inferencing (   )
    - Valid & Invalid Arguments
    - Sound and Unsound Arguments
- Modus Ponens and Modus Tollens
- Additional Valid Argument Forms: Rules of Inference
- Valid Arguments & Proof by Contradiction
- Sound and Unsound Arguments
- Common Logical Fallacies
    - Circular Reasoning: The conclusion is used as a premise
    - Ambiguous Premise: Premise is unclear or multiple meanings
    - Jump to Conclusion: Conclusion drawn without sufficient evidence
    - Converse Error: Assuming "if A, then B" implies "if B, then A"
    - Inverse Error: Assuming "if A, then B" implies "if not A, then not B"
- Summary of Rules of Inference

- **[Self] Application: Digital Logic Circuits**:
    - Black Boxes and Gates
        -
    - The Input/Output Table for a Circuit
    - The Boolean Expression Corresponding to a Circuit
    - The Circuit Corresponding to a Boolean Expression
    - Finding a Circuit That Corresponds to a Given Input/Output Table
    - Simplifying Combinational Circuits; NAND and NOR Gates

Chapter 3: The Logic of Quantified statements

- **Predicates and Quantified Statements (1)**:
    - Predicates, domains, truth sets
    - The Universal and Existential Quantifiers
    - Formal versus Informal Language
    - Universal Conditional Statements
    - Equivalent Forms of Universal and Existential Statements
    - Bound Variables and Scope
    - Implicit Quantification
- **Predicates and Quantified Statements (2)**
    - Negations of Quantified Statements
        - Negations of Universal Conditional Statements
        - Negations of Existential Conditional Statements
    - The Relation among ∀, ∃, ∧, and ∨
    - Vacuous Truth of Universal Statements ( )
    - Variants of Universal Conditional Statements
    - Necessary and Sufficient Conditions, Only-If
- **Arguments with Quantified Statements**:
    - Universal Instantiation
    - Existential Instantiation (from chapter 4)
    - Universal Modus Ponens
    - Use of Universal Modus Ponens in a Proof
        - Example: Area of a triangle
    - Universal Modus Tollens
        - Example: Sum of a rational and an irrational number.
        - Counter Example: Divisibility by 4 and 2.
    - Using Diagrams to Test for Validity
    - **[Self]** Creating Additional Forms of Argument
    - **[Self]** Remark on the Converse and Inverse Errors

Chapter 4: Elementary Number Theory and Methods of Proof

Elementary Number Theory

- **Set of Numbers and Their Properties**
    - Natural Numbers (***N***)

        - Even, Odd, Prime, and Composite numbers
        - Divisibility
            - Definition of Divisibility (divisible, multiple, factor, divides),
            - Factors of Natural Numbers
            - Proving Properties of Divisibility

                - Divisibility is reflexive and anti-symmetric, and transitive
                - Divisibility by a Prime
            - The Unique Factorization of Integers Theorem
            - Quotient-Remainder Theorem
            - *div* and *mod*
    - Integers (***Z***)
    - Rational Numbers (***Q***)
        - Integers are rational
        - Equivalency
        - Closed under +. -. *. /
        - Density
    - Real Numbers (***R***)
        - Absolute Value
            - Triangle Inequality
        - Floor and Ceiling
        - Density

Methods of Proof

- **Mathematical Proofs**
    - Deductive Proofs
        - Proving Existential Statements
            - Proving by Examples
            - Existential Instantiation
        - Proving Universal Statements
            - Method of Exhaustion
            - Universal Instantiation (and how it differs from the existential instantiation)
        - Disproving Universal Statements by Counterexample
        - Getting Proofs Started (proof template)
        - Examples
            - For all integers r and s, if r is even and s is odd then 3r +2s is even
    - Inductive Proofs (Covered in Chapter 5)
- **Writing of a Proof**
    - Invalid arguments might still have the correct conclusion
    - Tips on Writing a Coherent Proof
        - Copy the statement of the theorem to be proved on your paper.

        - Clearly mark the beginning of your proof with the word **Proof**.
        - Make your proof self-contained.
        - Write your proof in complete, grammatically correct sentences.
        - Keep your reader informed about the status of each statement in your proof.
        - Give a reason for each assertion in your proof.
        - Include the “little words and phrases” that make the logic of your arguments clear (e.g. *hence, therefore, so*)
        - Display equations and inequalities.
    - Common Mistakes
        - Arguing from examples
        - Using the same letter to mean two different things
        - Jumping to a conclusion
        - Assuming what is to be proved
        - Confusion between what is known and what is still to be shown
        - Use of <u>any</u> when the correct word is <u>some</u> (instead of) or vice versa
    - Examples:
        - There is no smallest positive rational number.
        - For all integers n, is odd.
- Conjecture
    - Goldbach's Conjecture
    - Twin Primes

Chapter 5: Sequences, Mathematical Induction, and Recursion

- **Sequences**:
    - Definition
        - Term
        - Index/Subscript
        - Upper and Lower Limits
    - Convert to Sum of Terms and Back

    - Finite and Infinite Sequences
    - Arithmetic and Geometric Sequences
    - Converging, Diverging, and Alternating Sequences
    - Summation Notation
    - Product Notation
    - Properties of Summations and Products
        - Changing the Lower/Upper Limits
        - Changing the Index Variable
        - Adding Two Sums
        - Multiplying Two Products
        - Factoring out a Constant
    - Factorial
            - Definition
            - 0!
- **[Self]** Sequences in Computer Programming
- **[Self]** Application: Algorithm to Convert from Base 10 to Base 2 Using Repeated Division by 2
- Pascal Triangle and Combinations
- **Mathematical Induction (1)**:
    - Deductive vs Inductive Logic
    - Proving Formulas: Principle of Mathematical Induction
    - Sum of the First n Integers
        - Carl Gauss's Method
    - Weak vs Strong Induction
    - Sum of a Arithmetic Sequence
    - Sum of a Geometric Sequence
- **Defining Sequences Recursively**:
    - Definition of Recursive Sequences
    - Examples of Recursively Defined Sequences
    - Recursive Definitions of Sum, Product, and Factorial
    - Recursive Definitions of Geometric and Arithmetic Sequences
    - Fibonacci Sequence
        - Golden Ratio
    - Recursive Programming
- **General Recursive Definitions and Structural Induction**:
    - Recursively Defined Sets:
        - Base
        - Recursion
        - Restriction (could sometimes be omitted using **iff** in Recursion)
        - Recursive Definitions for:
            - Numerical Expressions
            - Parenthesis Structures
    - Using Structural Induction to Prove Properties about Recursively Defined Sets
    - Recursive Functions

Chapter 6: Set theory

- **Set Theory: Definitions and the Element Method of Proof:**
    - Definition
    - Membership Relation
    - Subsets, Proper Subsets
        - Proof and Disproof for Subsets
    - Set Equality Using Subset Relation
    - The Universal Set
        - Set of All Sets?
    - The Empty Set
    - Venn Diagrams
    - Connection between Logic and Sets
    - Operations on Sets
        - Union
        - Intersection
        - Difference
        - Complement
    - Power Sets
    - Disjoint Sets

        - Partitions of Sets
- **Subsets**
    - Properties of Subsets
        - Inclusion of Intersection
        - Inclusion in Union
        - Transitivity
        - Anti-Symmetry
    - Proving Subset Relations
        - Using Implications
        - Set Equality
- **Size (Cardinality) of a Set**
    - The Notation
    - The Size of Union and Intersection
    - Size of the Power Set
    - Size of the Complement
- **Intervals**
    - Definition And Notation
    - Diagram
    - Logical Form
    - Open and Close Intervals
- **Set Identities**

- The Connection Between Boolean Algebra and Set Theory

- **Algebraic Proofs**:
    - “Algebraic” Proofs of Set Identities
- [Extra]
    - Representation Power of Sets
        - Ordered Pairs
        - Natural Numbers

- Infinity Hotel

- Russel's Paradox:

Chapter 7: Properties of Functions

- **Functions Defined on General Sets**:
    - Definition
        - Domain
        - Codomain/Range
    - Well Defined Functions
    - Function Diagrams
    - Functions as Ordered Pairs
    - Equality of Functions
        - Based on the Formula
        - Based on the Ordered Pairs
    - Prove/Disprove a Relation is a Function
    - Examples
        - Non-numerical
            - IsPalindrome(string) -> True/False
            - BirthDate(person) -> date
            - Mother(person) -> person
            - ~~Child(person) -> person~~ (not a function)
        - Numerical
            -
            -
            -
            -
            - Boolean Functions
                - Propositional Logic

            - Conjunction, Disjunction and Negation as Boolean Functions

- **One-to-One, Onto, and Inverse Functions**:
    - One-to-One (Injective) Functions

        - Definition
        - Injections in Diagrams
        - In Ordered Pairs
        - In x-y Plots
        - Inverse
        - Prove/Disprove a Function is an Injection
        - Injections have an inverse function
            - Computing the Inverse
    - Onto (Surjective) Functions

        - Definition
        - Surjections in Diagrams
        - In x-y Plots
        - Prove/Disprove a Function is an Surjections
    - One-to-One and Onto (Bijective) Functions
        - Comparing Set Sizes
        - Examples
- **Structure Preservation and Isomorphism**
    - Isomorphism: Bijection which preserves operations
        - Street Maps
        - Boolean Functions
        - Linear Equations
    - Not all bijections are isomorphism
    - Importance of Isomorphism in Identifying 'Sameness'
- Map of everything defined above

- **Composition of Functions**:
    - Definition and Examples
    - Properties
        - Not Commutative
        - Transitive
    - Composition of One-to-One Functions
    - Composition of Onto Functions
    - Inverse of a Composition

Chapter 8: Properties of Relations

- **Definition**
    - Ordered Pair
    - Visual Diagram & Plot
    - One to one, one to many, many to one, many to many relations
- **Relations on Sets**
    - Examples of Relations
        - Simple Ordered Pair
        - Circles

        - Inequalities
        - Between
        - Subsets
        - Divisibility
        - Cousin/Friend
        - Similarity/Closeness
    - The Inverse of a Relation; Directed Graph of a Relation
    - n-ary Relations and Relational Databases
- **Reflexivity, Symmetry, and Transitivity**
    - Reflexivity
    - Symmetry
    - Transitivity
- **Equivalence Relations**
    - Definition of an Equivalence Relation
    - Graph Representation
    - Equivalence Classes of an Equivalence Relation and Representatives
    - Examples
        - Sibling
        - Rational Numbers
        - Congruence Relation
    - Counter Examples
        - Coworker
        - Subset
        - Parallel Lines
    - Modular Arithmetic
        - Congruency relations and modulo
        - Properties of Congruency
            - Transitivity
            - Closed under +, -, *
- **Partial order relations**
    - Definition
        - Reflexive
        - Anti-Symmetric
        - Transitive
    - Examples:
        - Inequality
        - Subset
        - Divisibility
    - Hasse Diagrams
    - Maximal and Minimal Elements
    - Greatest and Least Elements
    - Totality and Total Orders

    - Dense Property
    - Dense Total Orders
    - Open and Close Intervals
    - Upper and Lower Bounds
        - Least Upper Bound
        - Greatest Lower Bound

Chapter 9: Counting and Probability

- **Counting & Size**
    - Lists, Sublists, and One-Dimensional Arrays
    - Permutations: Arrangements of objects where order matters
        - With or Without Replacement
    - Combinations: Selections of objects where order doesn't matter
    - Pigeonhole Principle
- **Possibility Trees and the Multiplication Rule**
    - Possibility Trees
    - The Multiplication Rule (AND)
        - Two Dice
    - Permutations
        - ~~Selected elements~~
        - Repeated Elements
        - Circular cases

Counting Elements of Disjoint Sets

- The Addition Rule: The Addition Rule (OR)

The Pigeonhole Principle

- Statement and Discussion of the Principle
- Generalized Pigeonhole Principle
- Pigeonhole Principle and injections
- One-to-One and Onto functions on Finite Sets
- Examples:
    - Socks in a Drawer
    - If 5 points are placed inside a square with side length 2, what is the minimum distance between two points?

Combinations

- ~~Combinations~~
- ~~Ordered and Unordered Selections~~
- ~~Relation between Permutations and Combinations~~
- ~~Permutation of a Set with Repeated Elements~~
- **Combinations with Repetition Allowed**
    - Equivalence Classes of Selections

        - The Formula and the Explanation of it
            - Example:
                - Distribute identical books to some friends
                - Roll 4 dic and note only the numbers rolled. How many distinct combinations of numbers are possible?

Identifying Order and Replacement

- Examples

    - Arrange the letters in the word GOOGLE
    - Draw 10 cards from a deck of cards
    - Set of all 10-letter English words
    - Select from a pile of coins, so the sum is $1 exactly

Introduction to Probability

- Definition
    - Sample Space
    - Event
    - Randomness
- Set Sizes
    - Infinite Sets
    - Real Numbers
    - 2D Cases and Area
- Probability in the Equally Likely Case, the Formula
- The true meaning of a probability number
    - The Gambler's Fallacy
- Probability with Sets:
    - Probability of the negation/complement
    - Probability of Intersection (AND)
    - Probability of Union (OR)
    - Probability of Subtraction

Binomial Theorem and Binary Outcomes

- Probability of an event occurring exactly r times in n trials:

Interesting Probability Examples

- Coin
- Dice
- Deck of Cards
- Shared Birthday
- Monty Hall Problem
- Flip a coin three times. What’s the probability of getting: Exactly 2 heads? At least 1 head?

- A family has two children, and you know one of them is a boy. What's the probability that both are boys?
- An email spam filter correctly identifies spam emails 95% of the time. If a user receives 10 emails in a day, what is the probability that the filter misclassifies at most two emails?

Probability Axioms and Expected Value:

- Probability Axioms
- Deriving Additional Probability Formulas
- Expected Value

[Extra] Chapter 10: Theory of Graphs and trees

- **Graphs**
    - History
    - Definition
        - Vertices and Edges
            - Degree of a Vertex
            - Isolated Vertices
        - Relations and Graphs
    - Undirected, Directed, Weighted Graphs
    - Representation
        - Set of Vertices & Set of Edges
        - Pairs of Vertices and Adjacent Vertices
        - Matices
    - Subgraphs

Walks, Trails, Paths, and Circuits:

- *Trail*: Walk with no repeated edge
- *Path*: Walk with no repeated vertex
- *Closed Walk*: Walk with the same start and end vertex
- *Circuit*: Closed Walk with no repeated edge
- *Simple Circuit*: Circuit with no repeated vertex

- **Connectedness**
    - Connected Vertices
    - Connected Graphs
    - Disconnected Graphs and Components
    - Euler Circuits
        - Degrees of Vertices
    - Euler Trails
    - Hamiltonian Circuits
        - Traveling Salesman Problem
- **Trees**
    - Definition
    - Properties of Trees
        - Number of Vertices and Edges
        - Leaves and Internal Vertices (Nodes)
    - Examples
        - Spanning Tree of a Graph
- **Rooted Trees**:
    - Definition
        - Root
        - Children
        - Parent
        - Height
    - Binary Trees
        - Example: Mathematical Expressions
    - Binary Search Trees
        - Insertion
        - Search

Student Learning Outcome(s):

- Critique a mathematical statement for its truth value, defend choice by formulating a mathematical proof or constructing a counterexample.
- Analyze and apply patterns of discrete mathematical structures to demonstrate mathematical thinking.

Office Hours:

F	3:00 PM - 4:00 PM	Zoom