## SECTION 10.1 PROBLEM SET: INTRODUCTION TO MARKOV CHAINS

1) Is the matrix given below a transition matrix for a Markov chain? Explain.

|  |  |
| --- | --- |
| a)  | b)  |

2) A survey of American car buyers indicates that if a person buys a Ford, there is a 60% chance that their next purchase will be a Ford, while owners of a GM will buy a GM again with a probability of .80. The buying habits of these consumers are represented in the transition matrix below.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  **Next Purchase**  |  |  |
|  |  |  | Ford | GM |  |
| **Present**  | Ford |  | .60 | .40 |  |
| **Purchase** | GM |  | .20 | .80 |  |

Find the following probabilities:

|  |  |
| --- | --- |
| a) The probability that a present owner of a Ford will buy a GM as his next car.  | b) The probability that a present owner of a GM will buy a GM as his next car.  |
| c) The probability that a present owner of a Ford will buy a GM as his third car. | d) The probability that a present owner of a GM will buy a GM as his fourth car. |

3) Professor Hay has breakfast at Hogee's every morning. He either orders an Egg Scramble, or a Tofu Scramble. He never orders Eggs on two consecutive days, but if he does order Tofu one day, then the next day he can order Tofu or Eggs with equal probability.

|  |  |
| --- | --- |
| a) Write a transition matrix for this problem.    | b) If Professor Hay has Tofu on Monday, what is the probability he will have Tofu on Tuesday? |
| c) If Professor Hay has Eggs on Monday, find the probability he will have Tofu on Wednesday. | d) If Professor Hay has Eggs on Monday, what is the probability he will have Tofu on Thursday? |

***SECTION 10.1 PROBLEM SET: INTRODUCTION TO MARKOV CHAINS***

4) A professional tennis player always hits cross-court or down the line. In order to give himself a tactical edge, he never hits down the line two consecutive times, but if he hits cross-court on one shot, on the next shot he can hit cross-court with .75 probability and down the line with .25 probability.

|  |  |
| --- | --- |
| a) Write a transition matrix for this problem.   | b) If the player hit the first shot cross-court, what is the probability that he will hit the third shot down the line?  |

5) The transition matrix for people voting for candidates from various political parties in an election year is given below. If a person votes for the candidate from one party in an election, that person may vote for the same party in the next election or may switch to vote for a candidate from another party in the next election. Democrats, Republicans, and Independents are denoted by the letters D, R, and I.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  | Next Election |  |
|  |  |  | D | R | I |  |
|  | D |  | .6 | .3 | .1 |  |
| This  | R |  | .3 | .6 | .1 |  |
| Election | I |  | .2 | .2 | .6 |  |

Assume there is an election every year so that the transition period is 1 year.

|  |  |
| --- | --- |
| a) Find the probability that a person who votes Democratic in the current election will vote Republican in the next election.  | b) Find the probability that a person who votes Democratic in the current election will vote Republican in the election two years from now. |
| c) Find the probability that a person who votes Republican in the current election will vote Independent in the election two years from now. | d) Find the probability that a person who votes Democratic in the current election will vote independent in the election three years from now.  |

## SECTION 10.2 PROBLEM SET: APPLICATIONS OF MARKOV CHAINS

Questions 1-2 refer to the following:

Reference: Bart Sinclair, Machine Repair Model. OpenStax CNX. Jun 9, 2005 [Creative Commons Attribution License 1.0](http://creativecommons.org/licenses/by/1.0). Download for free at [http://cnx.org/contents/56f1bed0-bd34-4c28-a2ec-4a3f9ded8e18@3](http://cnx.org/contents/56f1bed0-bd34-4c28-a2ec-4a3f9ded8e18%403). This material has been modified by Roberta Bloom, as permitted under that license.

A Markov chain can be used to model the status of equipment, such as a machine used in a manufacturing process. Suppose that the possible states for the machine are

 Idle & awaiting work (I) Working on a job/task (W) Broken (B) In Repair (R)

The machine is monitored at regular intervals to determine its status; for ease of interpretation in this problem, we assume the status is monitored every hour. The transition matrix is shown below. 

|  |  |
| --- | --- |
| 1. Use the transition matrix to identify the following probabilities concerning the state of the machine one hour from nowa) Find the probability that the machine is working on a job one hour from now if the machine is idle now.b) Find the probability that the machine is idle one hour from now if the machine is working on a job now.c) Find the probability that the machine is working on a job one hour from now if the machine is being repaired now.d) Find the probability that the machine is being repaired in one hour if it is broken now. | 2. Perform the appropriate calculations using the transition matrix to find the following probabilities concerning the state of the machine three hours from now.a) Find the probability that the machine is working on a job three hours from now if the machine is idle now.b) Find the probability that the machine is idle three hours from now if the machine is working on a job now.c) Find the probability that the machine is working on a job three hours from now if the machine is being repaired now.d) Find the probability that the machine is being repaired in three hours if it is broken now. |

***SECTION 10.2 PROBLEM SET: APPLICATIONS OF MARKOV CHAINS***

**Questions 3-4 refer to the following description of how a Markov chain might be used to “train” a computer to generate music.**

Teaching a computer music theory so that it can create music would be an extremely tedious task.
You would have to teach chord structure, different musical styles, and so on. What if you could give the program examples of pieces you considered to be music and ask it, “write something like that for me.” This is essentially how our Markov chain would work. The principle behind Markov chains in music is to generate a probability table to determine what note should come next. By feeding the program an example piece of music, the program can analyze the piece and create a probability table to determine which notes are more likely follow a given note. With the probability transition matrix one can generate random notes that still has some musical structure to it. By constructing a similar matrix for beats or note durations, one can complete a Markov chain model for music generation.

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The transition matrix below provides an example. The states are the notes A, A#,B,C,D,E,F,G,G#.
The matrix shows the probability of the next note (column state), given the current note (row state).

To generate computer created music, a computer program would randomly select the next note based on the previous note and the probabilities given in the transition matrix.

 

|  |  |
| --- | --- |
| 3) Find the probability for the next note, given the current notea. P(next note is A | current note is G) = \_\_\_\_\_\_b. P(next note is G | current note is A) = \_\_\_\_\_\_c. P(next note is C | current note is F) = \_\_\_\_\_\_d. P(next note is E | current note is D) = \_\_\_\_\_\_ | 4a) If the current note is A# (A-sharp) what does the transition matrix tell us about the next note?4b) If the next note is F, what do we know about the current note.  |

## SECTION 10.2 PROBLEM SET: APPLICATIONS OF MARKOV CHAINS

Question 5 refers to the following:

Markov chains play an important role in online search.

“PageRank is an algorithm used by Google Search to rank websites in their search engine results. PageRank was named after Larry Page, one of the founders of Google. PageRank is a way of measuring the importance of website pages”
Source: https://en.wikipedia.org/wiki/PageRank under the Creative Commons Attribution-ShareAlike License;

The theory behind PageRank is that pages that are linked to more often are more important and useful; identifying those that are linked to more often about a topic helps identify the pages that should be presented as most pertinent in a search.

In real world search, there are thousands or millions of pages linking together, resulting in huge transition matrices. Because of the size and other properties of these matrices, the mathematics behind PageRank is more sophisticated than the small example we examine here with only four websites. However our example is adequate to convey the main concept of PageRank and its use in search algorithms.

It should be noted that real world search algorithms, PageRank or similar Markov chain ranking schemes are only one of a variety of methods used.

Suppose we have 4 webpages that contains links to each other. We call the pages A, B, C, D.

* From page A, 30% of people link to page B, 50% link to page C, and 20% link to page D
* From page B, 50% of popele link to page A and 50% link to page D
* From page C, 10% of people link to page B, 70% link to page C, and 20% link to page D
* From page D, 20% of people link to page A, 40% to page B, 10% to page C, and 30% link to page D

(In this example, when a page links to itself, it means that a person viewing the page stays at that page and does not link to another page.)

|  |  |
| --- | --- |
| a) Write the transition matrix T  | b) Find the probability that a person viewing page C will link to page D next.  |
| c) Find the probability that a person viewing page C will view page D after two links | d) Find the probability that a person viewing page C will view page D after **three** links |
| e) Find the probability that a person viewing page C will stay at page C and not link to any other page next.  | f) Find the probability that a person viewing page C will view page A next  |

## SECTION 10.3 PROBLEM SET: REGULAR MARKOV CHAINS

1) Determine whether the following matrices are regular Markov chains.

|  |  |
| --- | --- |
| a)  | b)  |
| c)  | d)  |

2) Company I and Company II compete against each other, and the transition matrix for people switching from Company I to Company II is given below.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  |  **TO** |  |  |
|  |  |  | Company I | Company II |  |
| **FROM** | Company I |  | .3 | .7 |  |
|  | Company II |  | .8 | .2 |  |

|  |  |
| --- | --- |
| a) If the initial market share is 40% for Company I and 60% for Company II, what will the market share be after 3 transitions?   | b) If this trend continues, what is the long range expectation for the market?  |

3) Suppose the transition matrix for the tennis player in Exercise 4 of the last section is as follows, where C denotes the cross-court shots and D denotes down-the-line shots.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  | **Next Shot** |  |  |
|  |  |  | C | D |  |
| **Previous Shot** | C |  | .9 | .1 |  |
|  | D |  | .7 | .3 |  |

|  |  |
| --- | --- |
| a) If the player hit the first shot cross-court, what is the probability he will hit the fourth shot cross-court? | b) Determine the long term shot distribution. |

***SECTION 10.3 PROBLEM SET: REGULAR MARKOV CHAINS***

4) Professor Hay never orders eggs two days in a row, but if he orders tofu one day, then there is an equal probability that he will order tofu or eggs the next day.

Find the following:

|  |  |
| --- | --- |
| a) If Professor Hay had eggs on Monday, what is the probability that he will have tofu on Friday? | b) Find the long term distribution for breakfast choices for Professor Hay. |

5) In a bikeshare program with 3 bike stations, A, B, and C, people can borrow a bicycle at one station and return it to the same station or either of the other two stations. The transition matrix is:

 T = 

Find the following:

|  |  |
| --- | --- |
| a) If a bicycle is initially at station A, what is the probability it will be at station C after 5 days? | b) If the initial distribution of bicycles is 50% at station A, 20% at station B, and 30% at station C, what will be the distribution after 2 days? After 5 days? |
| c) What will be the eventual long term distribution of bicycles at the stations? | d) If initially the distribution of bicycles at the stations was evenly distributed with one third of the bicycles at each station, will the eventual long term distribution be different than if the initial distribution is as given in part (c)? |

***SECTION 10.3 PROBLEM SET: REGULAR MARKOV CHAINS***

6) Suppose that a country has 3 political parties: the Conservative (C), Liberal (L), and National (N) parties. If a person votes for the candidate from one party in an election, that person may decide to vote for the same party in the next election or may switch to vote for a candidate from another party in the next election. The transition matrix is:

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | NEXT ELECTION |  |
|  |  |  | C | L | N |  |
|  | C |  | .5 | .4 | .1 |  |
| THIS  | L |  | .3 | .4 | .3 |  |
| ELECTION | N |  | .2 | .2 | .6 |  |

Assume there is an election every year, so the transition time is one year. Find the following.

|  |  |
| --- | --- |
| a) If a person voted for the Liberal party in this election, find the probability that the person votes for the National party in the next election.  | b) If a person voted for the National party in this election, find the probability that the person votes for the Conservative party in the election two years from now.  |
| c) If in this election Conservatives received 25% of the votes, Liberals 30% of the votes, and Nationals the remaining 45% of the votes, what is the predicted distribution for the next election?  | d) Assuming the current distribution from part (c), what will the distribution be in the election two years from now? |
| e) Assuming the current distribution from part (c), what will the distribution be in the election three years from now? | f) Determine the long term distribution. |

***SECTION 10.3 PROBLEM SET: REGULAR MARKOV CHAINS***

**Question 7 refers to the following:**

Markov chains play an important role in online search.

“PageRank is an algorithm used by Google Search to rank websites in their search engine results. PageRank was named after Larry Page, one of the founders of Google. PageRank is a way of measuring the importance of website pages”
Source: https://en.wikipedia.org/wiki/PageRank under the Creative Commons Attribution-ShareAlike License;

The theory behind PageRank is that pages that are linked to more often are more important and useful; identifying those that are linked to more often about a topic helps identify the pages that should be presented as most pertinent in a search.

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It should be noted that real world search algorithms, PageRank or similar Markov chain ranking schemes are only one of a variety of methodsused.

Suppose we have 4 webpages that contains links to each other. We call the pages A, B, C, D.

* From page A, 30% of people link to page B, 50% link to page C, and 20% link to page D
* From page B, 50% of popele link to page A and 50% link to page D
* From page C, 10% of people link to page B, 70% link to page C, and 20% link to page D
* From page D, 20% of people link to page A, 40% to page B, 10% to page C, and 30% link to page D

 (In this example, when a page links to itself, it means that a person viewing the page stays at that page and does not link to another page.)

|  |  |
| --- | --- |
| a)Write the transition matrix. | b) T is a 4x4 matrix, n= 4 states. Use the formula m = ( n−1)2 + 1 to find the highest power m that we need to check to determine if T is a regular Markov chain. |
| c) Is this a regular Markov chain? Explain how you determined that.  | d) Find the equilibrium vector and write a sentence summarizing the long term distribution of visits to these sites based on this model. |
| e) In the equilibrium vector, the state with the highest probability has the highest “Page-rank” and as the probabilities decrease, the ranking decreases. Indicate the order of the ranking, from highest Page rank to lowest Page rank, of these 4 pages. |

## SECTION 10.4 PROBLEM SET: ABSORBING MARKOV CHAINS

1) Given the following absorbing Markov chain.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | S1 | S2 | S3 | S4 |  |
|  | S1 |  | 1 | 0 | 0 | 0 |  |
| T = |  S2 |  | .1 | .4 | .2 | .3 |  |
|  | S3 |  | 0 | 0 | 1 | 0 |  |
|  | S4 |  | .4 | 0 | .2 | .4 |  |

|  |  |
| --- | --- |
| a) Identify the absorbing states.  | b) Write the solution matrix. |
| c) Starting from state 4, what is the probability of eventual absorption in state 1? | d) Starting from state 2, what is the probability of eventual absorption in state 3? |

2). Two tennis players, Andre and Vijay each with two dollars in their pocket, decide to bet each other $1, for every game they play. They continue playing until one of them is broke.

|  |  |
| --- | --- |
| a) Write the transition matrix for Andre. | b) Identify the absorbing states.  |
| c) Write the solution matrix. | d) At a given stage if Andre has $1, what is the chance that he will eventually lose it all? |

3) Repeat the previous problem, if the chance of winning for Andre is .4 and for Vijay .6.

|  |  |
| --- | --- |
| a) Write the transition matrix for Andre. | b) Write the solution matrix. |
| c) If Andre has $3, what is the probability that he will eventually be ruined? | d) If Vijay has $1, what is the probability that he will eventually triumph? |

***SECTION 10.4 PROBLEM SET: ABSORBING MARKOV CHAINS***

4) Repeat problem 2, if initially Andre has $3 and Vijay has $2.

|  |  |
| --- | --- |
| a) Write the transition matrix. | b) Identify the absorbing states.  |
| c) Write the solution matrix. | d) If Andre has $4, what is the probability that he will eventually be ruined? |

5) The non-tenured professors at a community college are regularly evaluated. After an evaluation they are classified as good, bad, or improvable. The "improvable" are given a set of recommendations and are re-evaluated the following semester. At the next evaluation, 60% of the improvable turn out to be good, 20% bad, and 20% improvable. These percentages never change and the process continues.

|  |  |
| --- | --- |
| a) Write the transition matrix. | b) Identify the absorbing states.  |
| c) Write the solution matrix. | d) What is the probability that a professor who is improvable will eventually become good? |

***SECTION 10.4 PROBLEM SET: ABSORBING MARKOV CHAINS***

**Questions 6 – 11 refer to the following:**In a professional certification program students take classes and then participate in an internships.
There are 4 states: taking classes (C), internship (I), drop out (D), and graduate (G).
If a student drops out they are never readmitted to the program.

Of those students currently taking classes, 70% have an internship the next year, 20% are still taking classes the next year, and 10% have dropped out by the next year.
 Of the students who are currently doing an internship, 65% graduate by the next year; 20% drop out by the next year, and 15% are still completing their internship the next year.

|  |  |
| --- | --- |
| 6) Write the transition matrix and indicate which are the absorbing states. | 7) If a student is taking classes now:a) find the probability that the student will graduate in 2 yearsb) find the probability that the student will be in the internship in 2 years.c) find the probability that the student will have dropped out by 2 years from now. |
| 8) Find the probability that a student currently doing an internship will eventually drop out. | 9) Find the probability that a student taking classes now will eventually graduate.  |
| 10) If 40% of students are currently taking classes and 60% of current students are doing internships, what is the eventual long term distribution of students for graduating versus dropping out? | 11) If 70% of students are currently taking classes and 30% of current students are doing internships, what is the eventual long term distribution of students for graduating versus dropping out? |

***SECTION 10.4 PROBLEM SET: ABSORBING MARKOV CHAINS***

12) A mouse is placed in the maze shown below, and it moves from room to room randomly.
From any room, the mouse will choose a door to the next room with equal probabilities.
 Once the mouse reaches room 1, it finds food and never leaves that room.
And when it reaches room 5, it is trapped and cannot leave that room.

What is the probability the mouse will end up in room 5 if it was initially placed in room 3?

 

13) In problem 12, what is the probability the mouse will end up in room 1 if initially placed in room 2?

## SECTION 10.5 PROBLEM SET: CHAPTER REVIEW

1) Is the matrix given below a transition matrix for a Markov chain? Explain.

a)  b) 

2) A survey of computer buyers indicates that if a person buys an Apple computer, there is an 80% chance that their next purchase will be an Apple, while owners of a Windows computer will buy an Windows computer again with a probability of .70. The buying habits of these consumers are represented in the transition matrix below.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  | **Next Purchase** |  |  |
|  |  |  | Apple | Windows |  |
| **Present** |  Apple |  | .80 | .20 |  |
| **Purchase** |  Windows |  | .30 | .70 |  |

a) Find the probability that a present owner of an Apple computer will buy a Windows computer as his next computer.

b) Find the probability that a present owner of an Apple computer will buy a Windows computer as his third computer.

c) Find the probability that a present owner of a Windows computer will buy a Windows computer as his fourth computer.

3) Professor Trayer either teaches Finite Math or Statistics each quarter. She never teaches Finite Math two consecutive quarters, but if she teaches Statistics one quarter, then the next quarter she will teach Statistics with a 1/3 probability.

a) Write a transition matrix for this problem.

b) If Professor Trayer teaches Finite Math in the Fall quarter, what is the probability that she will teach Statistics in the Winter quarter.

c) If Professor Trayer teaches Finite Math in the Fall quarter, what is the probability that she will teach Statistics in the Spring quarter.

4) Determine whether the following matrices are regular Markov chains.

a) b)

5) The transition matrix for switching academic majors each quarter by students at a university is given below, where Science, Business, and Liberal Arts majors are denoted by S, B, and A, respectively.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | **TO** |  |  |
|  |  |  | S | B | A |  |
|  | S |  | .6 | .3 | .1 |  |
| **FROM** | B |  | .1 | .7 | .2 |  |
|  | A |  | .1 | .1 | .8 |  |

a) Find the probability of a science major switching to a business major during their first quarter.

b) Find the probability of a business major switching to a Liberal Arts during their second quarter.

c) Find the probability of a science major switching to Liberal Artsr during their third quarter.

***SECTION 10.5 PROBLEM SET: CHAPTER REVIEW***

6) John Elway, the football quarterback for the Denver Broncos, called his own plays. At every play he had to decide to either pass the ball or hand it off. The transition matrix for his plays is given in the following table, where P represents a pass and H a handoff.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  | **Next Shot** |  |  |
|  |  |  | P | H |  |
| **Previous** | P |  | .6 | .4 |  |
| **Shot** | H |  | .8 | .2 |  |

a) If John Elway threw a pass on the initial play, what is the probability that he handed handoff on the two plays later?

 b) Determine the long term play distribution.

7) Company I, Company II, and Company III compete against each other, and the transition matrix for people switching from company to company each year is given below.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | **TO** |  |  |
|  |  |  | I | II | III |  |
|  | I |  | .6 | .2 | .2 |  |
| **FROM** | II |  | .3 | .5 | .2 |  |
|  | III |  | .3 | .3 | .4 |  |

a) If the initial market share is 20% for Company I, 30% for Company II and 50% for Company III, what will the market share be after the next year?

b) If this trend continues, what is the long range expectation for the market?

8) Given the following absorbing Markov chain.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | S1 | S2 | S3 | S4 |  |
|  | S1 |  | 1 | 0 | 0 | 0 |  |
| T = | S2 |  | 0 | 1 | 0 | 0 |  |
|  | S3 |  | .2 | .3 | .4 | .1 |  |
|  | S4 |  | .4 | .1 | .1 | .4 |  |

a) Identify the absorbing states.

b) Write the solution matrix.

c) Starting from state 4, what is the probability of eventual absorption in state 1?

d) Starting from state 3, what is the probability of eventual absorption in state 2?

9) A mouse placed in the maze moves from room to room randomly. From any room, the mouse will choose a door to the next room with equal probabilities. Once it reaches room 1, it finds food and never leaves that room. And when it reaches room 6, it is trapped and cannot leave that room.
What is the probability that the mouse will end up in room 1 if it was initially placed in room 3?

 

10) What is the probability that the mouse will end up in room 6 if it was initially in room 2?