First Name
Last Name
Campus ID

PHYS 4A – Spring 2024 – Final
2 Hours – Scientific calculator allowed
Lecture Notes, Books, Mobile Phones, Tablets, or Laptops are not allowed.

Question:	1	2	3	4	5	Total
Points:	5	10	10	5	10	40
Bonus Points:	2	2	2	0	2	8
Score:						

1. A rock is dropped vertically from a cliff and falls under the influence of gravity. A second rock is released 1.00 s later. *Ignore air friction.*

   (a) (5 points) How many seconds after the *first* rock is dropped, will the distance between two rocks be equal to 10.0 m?

   Solution: At the time the distance between the two rocks is equal to 10.0 m we can write using $\Delta t = 1.00\,\mathrm{s}$ and $d = 10.0\,\mathrm{m}$:

   $$\frac{1}{2}gt^2 = \frac{1}{2}g(t-\Delta t)^2 + d$$

   where the left-hand side is the distance traveled by the first rock and the right-hand side is 10.0 m plus the distance traveled by the second rock which started Δt later. Solving for t we get:

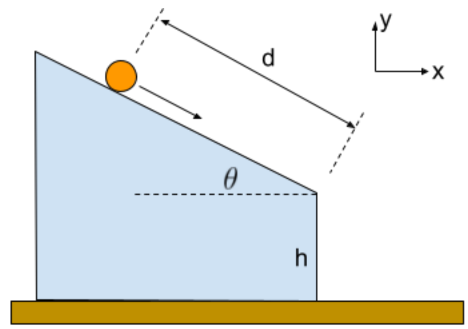
   $$0 = -gt\Delta t + \frac{1}{2}g(\Delta t)^2 + d \quad \Rightarrow \quad t = \frac{g(\Delta t)^2/2 + d}{g\Delta t} = \frac{\Delta t}{2} + \frac{d}{g\Delta t} = 1.52\,\mathrm{s}$$

   (b) (2 points (bonus)) Calculate the *relative velocity* of the first rock to the second rock when they are 10.0 m apart.

   Solution: The first rock will have a velocity of $v_1 = gt = 14.9\,\mathrm{m/s}$ and the second rock will have a velocity of $v_2 = g(t-\Delta t) = 5.10\,\mathrm{m/s}$. Therefore the relative velocity of the first rock compared to the second rock will be:

   $$v_{1/2} = v_1 - v_2 = 9.80\,\mathrm{m/s}$$

2. A uniform cylinder of mass $m = 2.00\,\text{kg}$ and radius R, initially at rest, *rolls* down a roof with a *rough surface*. The inclination is $\theta = 30.0°$ and the total distance is $d = 20.0\,\text{m}$. ($I_{\text{CM}} = mR^2/2$) *Hint: You don't need the value of R for the calculations below.*



(a) (3 points) Calculate the change in potential energy as the cylinder travels the distance d on the roof.

Solution: The potential energy change will be equal to $\Delta U = mg\Delta y = -196\,\text{J}$

(b) (4 points) Calculate the total kinetic energy, the rotational kinetic energy, and the translational kinetic energy of the cylinder as it's leaving the edge of the roof.

Solution: As this is a rolling motion we have $\omega = v_{\text{CM}}/R$. The total kinetic energy is due to the potential energy change where energy is conserved $E_i = E_f$:

$$E_i = mgd\sin\theta \quad \text{and} \quad E_f = \frac{1}{2}mv_{\text{CM}}^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{v_{\text{CM}}}{R}\right)^2$$

with the first term in E_f being the translational kinetic energy and the second term being the rotational kinetic energy. This gives us after simplification:

$$mgd\sin\theta = \frac{3}{4}mv_{\text{CM}}^2 \quad \Rightarrow \quad v_{\text{CM}} = \sqrt{\frac{4}{3}gd\sin\theta} = 11.437\,\text{m/s}.$$

Hence $K_{\text{rot.}} = 65.0\,\text{J}$ and $K_{\text{trans.}} = 131\,\text{J}$. From this, $K_{\text{tot}} = 196\,\text{J}$.

(c) (3 points) What is the velocity vector $\vec{\mathbf{v}}_{\mathrm{CM}}$ of the center of mass of the cylinder as it's leaving the edge of the roof in the coordinate frame shown on the graph?

Solution: The velocity will be given by:

$$\vec{\mathbf{v}}_{\mathrm{CM}} = \begin{pmatrix} v_{\mathrm{CM}} \cos\theta \\ -v_{\mathrm{CM}} \sin\theta \end{pmatrix} = \begin{pmatrix} 9.90\,\mathrm{m/s} \\ -5.72\,\mathrm{m/s} \end{pmatrix}$$

(d) (2 points (bonus)) If the edge of the roof is $h = 8.00\,\mathrm{m}$ above the ground, how far from the building will the cylinder fall?

Solution: We first calculate the velocity of the cylinder on impact. Using $v_f^2 = v_i^2 + 2a(x_f - x_i)$ for the y-component of the velocity calculated above:

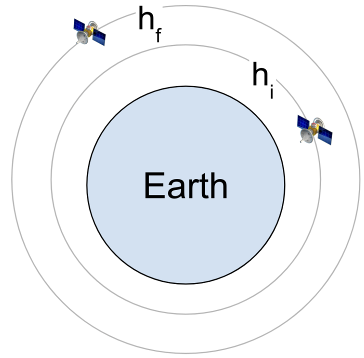
$$v_{yf}^2 = (-5.72\,\mathrm{m/s})^2 - 2 \times 9.81\,\mathrm{m/s^2} \times (-8.00\,\mathrm{m}) \quad \Rightarrow \quad v_{yf} = -13.77\,\mathrm{m/s}$$

we can then calculate the time it takes for the impact:

$$v_f = v_i + at \quad \Rightarrow \quad -13.8\,\mathrm{m/s} = -5.72\,\mathrm{m/s} - 9.81\,\mathrm{m/s^2} \times t \quad \Rightarrow \quad t = 0.821\,\mathrm{s}$$

Therefore the distance is $v_x \times t = 8.13\,\mathrm{m}$

3. A satellite of mass $m = 800\,\mathrm{kg}$ is orbiting the earth at an altitude of $h_i = 600\,\mathrm{km}$ in a circular orbit. We want to move this satellite to a circular orbit at an altitude of $h_f = 4300\,\mathrm{km}$. This is a two-stage process with a first burn to reach the transfer orbit and a second burn to reach the final orbit. *Remember:* $R_E = 6380\,\mathrm{km}$



(a) (4 points) What is the total amount of energy injected to do this?

Solution: The total energy injection is the difference in energies between the two circular orbits:

$$E_1 = -\frac{GM_E m_s}{2(R_E + h_1)} \quad \text{and} \quad E_2 = -\frac{GM_E m_s}{2(R_E + h_2)}$$

with

$$\Delta E = E_2 - E_1 = -\frac{GM_E m_s}{2R_E}\left(\frac{R_E}{R_E + h_2} - \frac{R_E}{R_E + h_1}\right)$$

This simplifies further to:

$$\Delta E = -\frac{m_s g R_E}{2}\left(\frac{6380}{10680} - \frac{6380}{6980}\right) = 7.93 \times 10^9\ \mathrm{J}$$

(b) (3 points) This manoeuver would require two steps with an elliptical transfer orbit inbetween. What is the length of the semi-major axis of this transfer orbit expressed in kilometers?

Solution: Using $2a = r_{\mathrm{min}} + r_{\mathrm{max}}$ we have:

$$2a = 6980\,\mathrm{km} + 10\,680\,\mathrm{km} \quad \Rightarrow \quad a = 8.83 \times 10^3\,\mathrm{km}$$

(c) (3 points) What is the orbital period of the satellite in the final position? Give your answer in minutes.

Solution: Using the formula:

$$T^2 = \left(\frac{4\pi^2}{GM}\right) a^3$$

we get

$$T = \sqrt{\frac{4\pi^2 (R_E + h_2)^3}{GM_E}} = 2\pi \sqrt{\frac{(R_E + h_2)^3}{gR_E^2}} = 1.0974 \times 10^4 \,\text{s} = 183 \text{ min}$$

(d) (2 points (bonus)) How long does the satellite stay on the transfer orbit before the second rocket burn? Give your answer in minutes.

Solution: Using the value of the semimajor axis calculated above, for the transfer orbit:

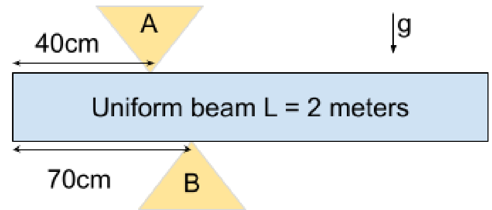
$$T^2 = \left(\frac{4\pi^2}{GM}\right) a^3 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{(R_E + h_2)^3}{gR_E^2}}$$

Plugging in $a = 8.83 \times 10^3$ km we obtain:

$$T = 138 \text{ min}$$

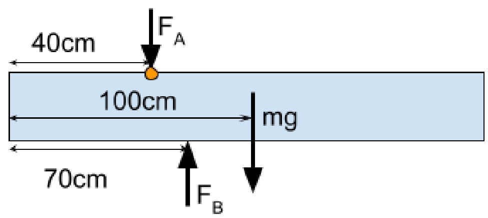
and as the satellite has to complete half the transfer orbit to get from perigee to apogee, the time required is 69 mins.

4. A *uniform* beam of mass $m = 100\,\mathrm{kg}$ and length $L = 2.00\,\mathrm{m}$ is being supported at two points $x_A = 40.0\,\mathrm{cm}$ and $x_B = 70.0\,\mathrm{cm}$ measured from one end of the beam.



(a) (2 points) Draw a free-body diagram of the beam and indicate and label all the forces.

Solution: Note here that for further calculations we chose the reference point indicated in the image below.



(b) (3 points) Calculate the forces F_A, and F_B that the supports **A** and **B** exert on the beam.

Solution: Calculating the torques around **A**, we get:

$$\sum \vec{\tau} = 0 = F_B \times 30.0\,\mathrm{cm} - mg \times 60.0\,\mathrm{cm}$$

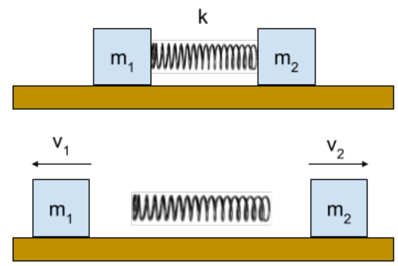
which gives us:

$$F_B = \frac{60.0}{30.0} \times mg = 1.96\,\mathrm{kN}$$

and using the equation $\Sigma \vec{\mathbf{F}} = 0$:

$$F_A + mg - F_B = 0 \quad \Rightarrow \quad F_A = 981\,\mathrm{N}$$

5. Two objects of masses $m_1 = 2.00\,\mathrm{kg}$ and $m_2 = 1.00\,\mathrm{kg}$ are connected with a massless spring of $k = 4000\,\mathrm{kg/s^2}$ and are resting on a *frictionless* surface. Initially, the spring is compressed by 10.0 cm and gets released.



(a) (3 points) What is the total energy of the system before the spring is released?

Solution: The total energy of the system is just the compression energy in the spring:

$$E_i = U_{\text{spring}} = \frac{1}{2}kx^2 = 20.0\,\mathrm{J}$$

(b) (2 points) What is the total momentum of the system before and after the spring is released?

Solution: The momentum before and after is zero.

(c) (5 points) What are the velocities of each block with respect to the table after the spring is released?

Solution: After the spring is released the energy of the spring will be given to the blocks as kinetic energy:

$$E_f = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

we also have the momentum conservation:

$$m_1v_1 + m_2v_2 = 0$$

Which gives us $v_2 = -v_1(m_1/m_2)$ When we plug this into the equation for energy:

$$E_f = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_1^2(m_1/m_2)^2 = \frac{1}{2}\frac{m_1m_2 + m_1^2}{m_2}v_1^2$$

Which gives us: $v_1 = -2.58\,\mathrm{m/s}$ and $v_2 = 5.16\,\mathrm{m/s}$.

(d) (2 points (bonus)) What is the velocity of the spring after the objects m_1 and m_2 have lost contact with the spring? *Hint: Consider the velocity of each end of the spring just before it loses contact with the objects.*

Solution: Just before the objects leave the spring, one end of the spring is travelling at v_1 and the other end is traveling at v_2 because it's expanding. This means the center point of the spring is moving at $(v_1 + v_2)/2$. Hence in this case it would be moving at $v_{\text{spring}} = 1.29\,\text{m/s}$